LECTURE 13

Computational Fluid Dynamics (CFD) wb1428

Mathieu Pourquie
m.j.b.m.pourquie@wbmt.tudelft.nl
http://www.ahd.tudelft.nl/∼mathieu/CFD.html

http://www.ahd.tudelft.nl
info for students
wb1428 Computational Fluid Dynamics

Fluid dynamics group
Stromingsleer
building part 5B
room 1-32
015-2782997

Last time:

- numerical diffusion
- numerical dispersion
- implicit methods

Today:

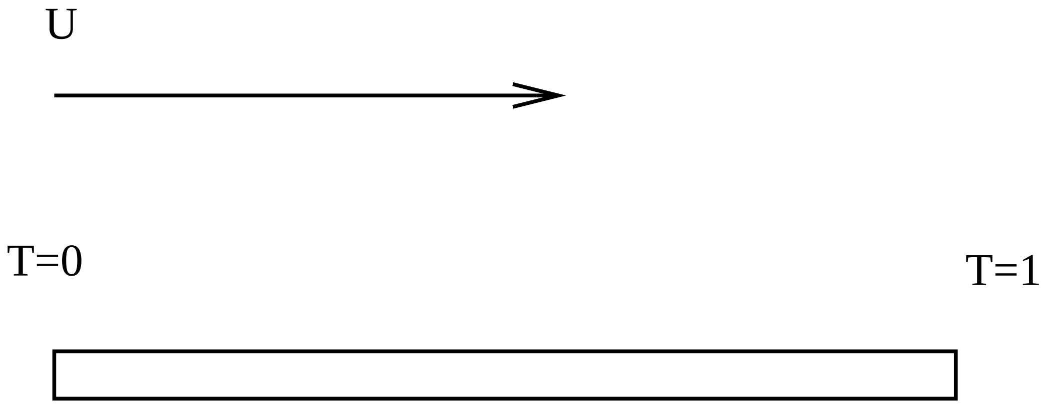
- combined advection-diffusion
- stability of one-step methods
- boundary conditions
- grid generation

DEMO Combined advection-diffusion (2)

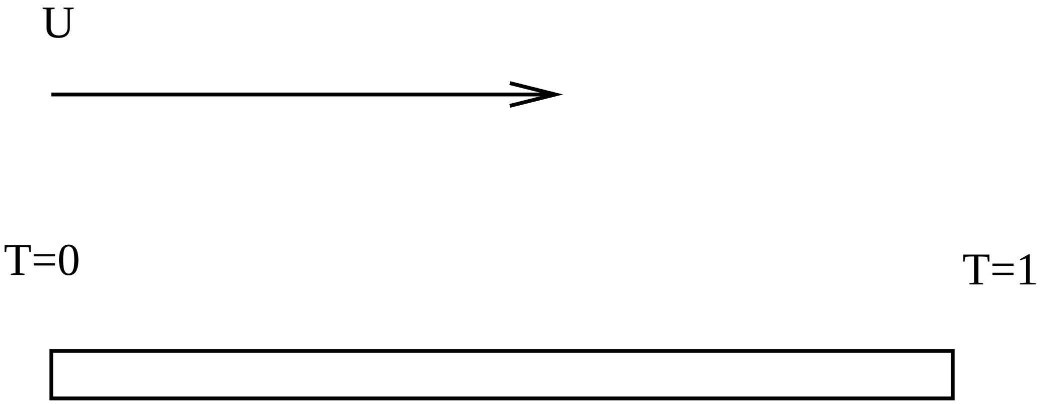
- relative importance advection-diffusion
- Peclet number
- numerical scheme: upwind and central differences

Some more facts about a model equation:

$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} + K\frac{\partial^2 C}{\partial x^2}$$

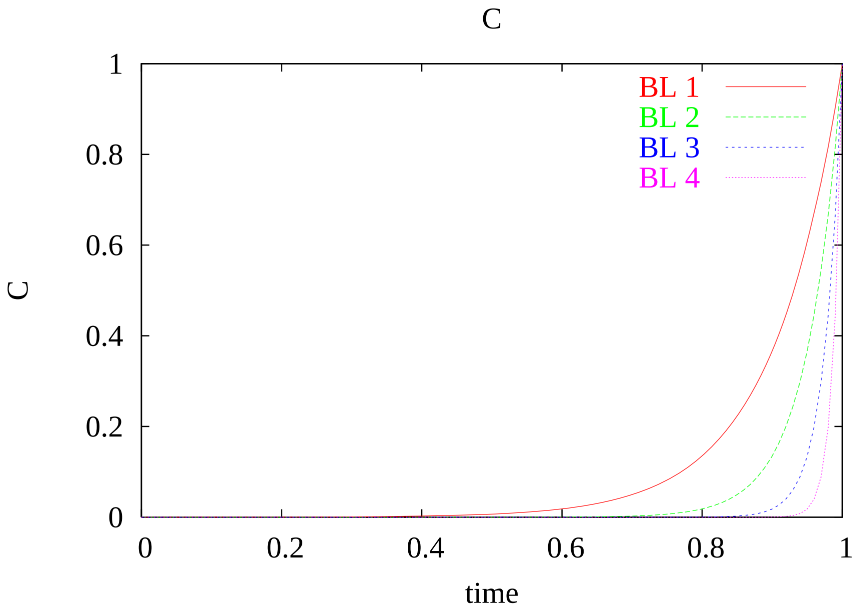


- BC
- only advection:
  - inflow only
- only diffusion or advection-diffusion:
  - inflow and outflow



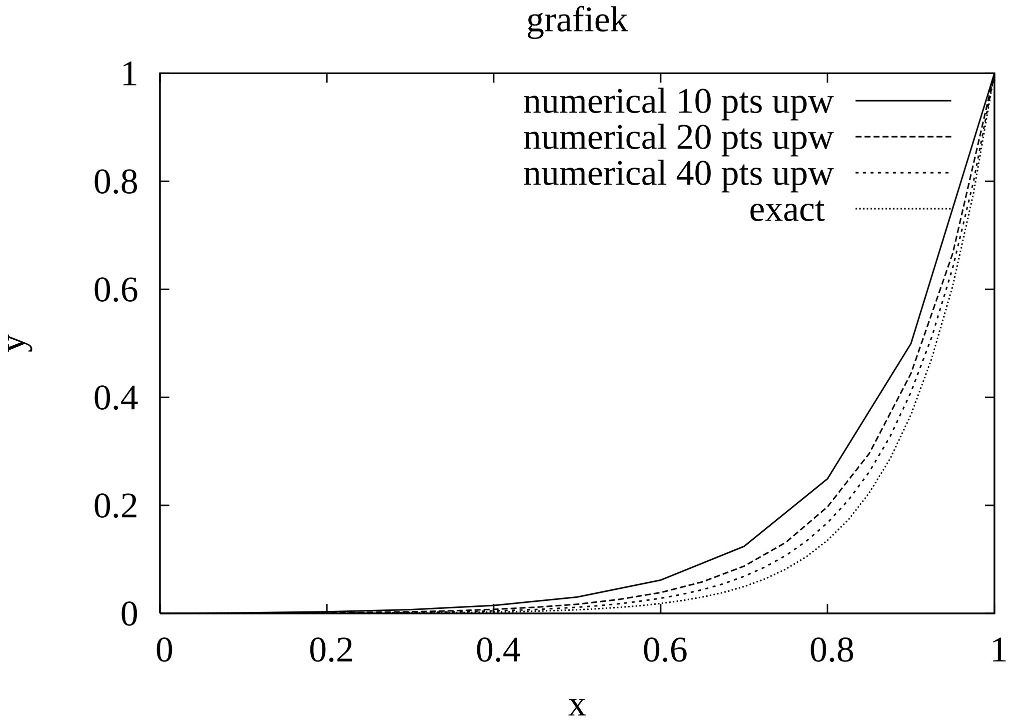
- flow to right
- diffusion both directions
- relative importance advection diffusion
- $Pe = UL/K$

- Exact solution,stationary, $\frac{exp(x*u/K)-1}{exp(u/K)-1}$
- exercise: $u = 1$ and $k = 0.1$
- $Pe = 10$
- Boundary Layer $Pe = UL/K$

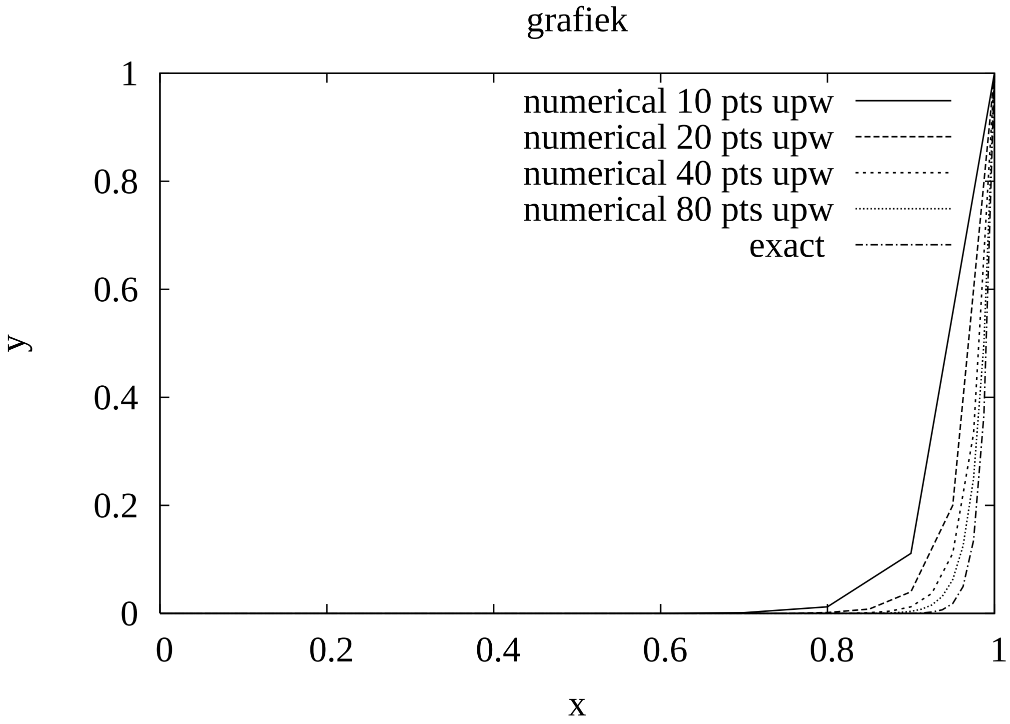


How well do we do numerically?

upwind advection, central diffusion cell Pe = 10



upwind advection, central diffusion cell Pe = 100



- numerical diffusion
- solution more accurate with more points
- upwind diffusion more important for high Pe

What about central differences?

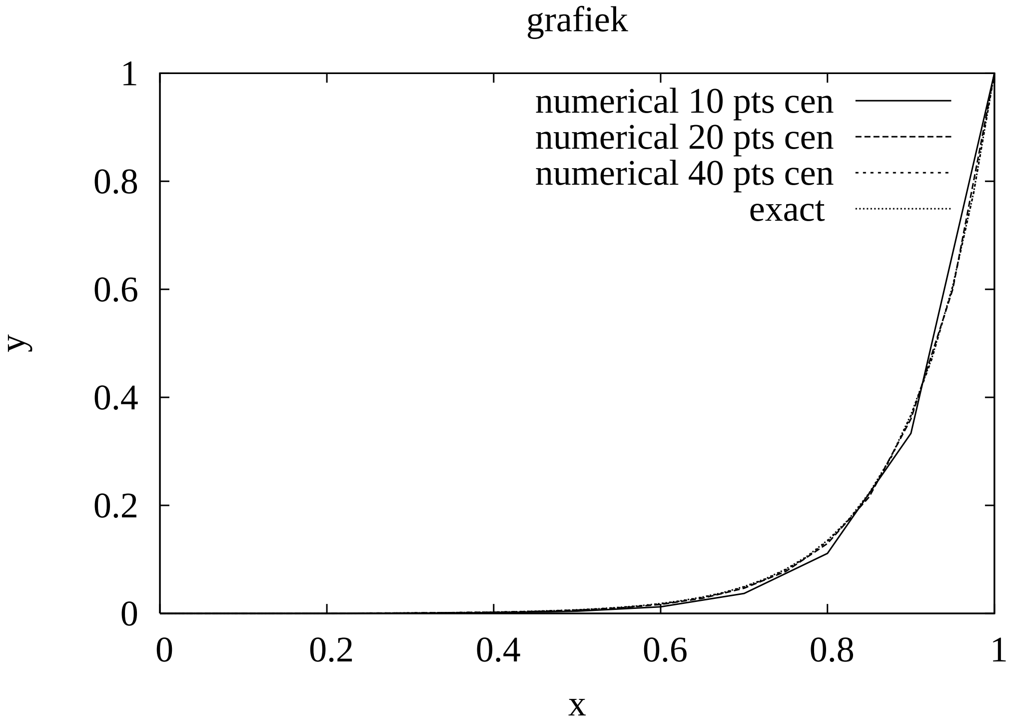
$$\frac{\partial C}{\partial x} \approx \frac{C(i+1) - C(i-1)}{2\Delta x}$$

NOTE a peculiarity with BC when $K = 0$!

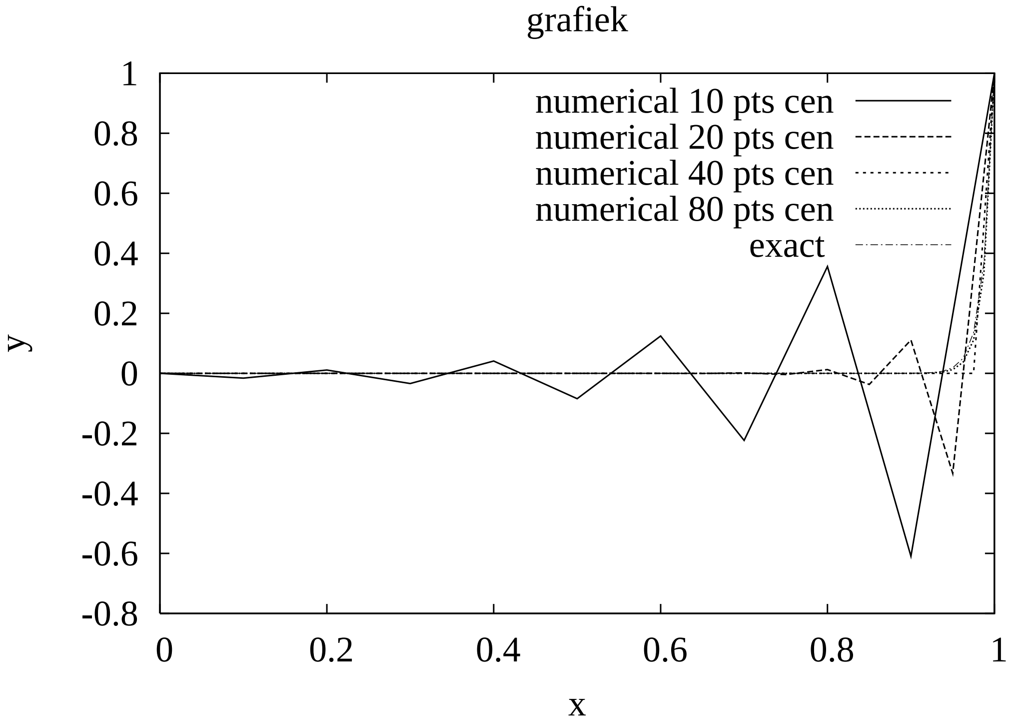
Only advection:

- advection equation: only BC at inflow
- central differences: also at RHS
- what will we take?

central advection, central diffusion cell $\mathrm{Pe} = 10$



central advection, central diffusion cell Pe = 100



- numerical wiggles
- solution more accurate with more points
- dispersion more important for high Pe

Central differences wiggle up to certain grid cell size

- $cellPe = u\Delta x/K$
- $cellPe > 2$ CDS wiggles
- $cellPe < 2$ CDS NO wiggles

NOTE: these wiggles are not the same as instability!

- Stability in multi-D problems
- stability for system of ODE

Remember multi-step methods? use analogon to 2nd order diff eqn
The discrete analog of ODE: Finite Difference Equations (FDE)

$$\frac{dC}{dt} + KC = 0$$
$$\frac{d^2C}{dt^2} + A\frac{dC}{dt} + B = 0$$

$$\begin{aligned} C^{n+1} + DC^n &= 0 \\ C^{n+1} + EC^n + FC^{n-1} &= 0 \end{aligned}$$

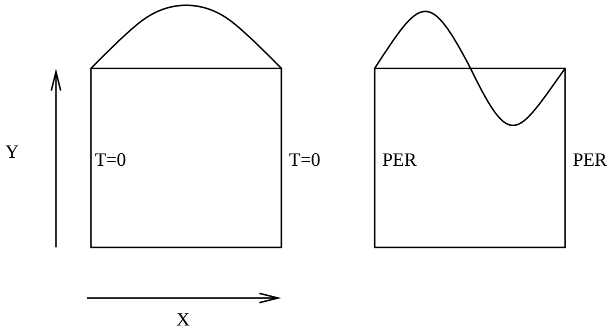
- A,B,D,E,F constant
- trial solution differential equation $Amp * e^{\lambda t}$
- nth order differential equation has n solutions
- trial solution difference equation $Amp * \lambda^n$
- nth order difference equation has n solutions

Another method is the matrix method:

- write ODE as system of first order ODE
- discretize time-derivative
- write system as $Cn = MCo$
- ev's M must be < 1
- ev is growth factor for eigenvector
- solution of the form λ^n corresponds to $Cn = \lambda Co$

- We now have PDE!
- Solution depends on BC
- only special cases can be treated
- special cases for very simple BC
- example: separation of variables

$$T(x,y) = X(x)Y(y)$$
$$X(x) = sin(k\pi x/L)$$
$$X(x) = sin(k2\pi x/L)$$



- likewise T(x,t) = X(x) f(t)

Stability analysis for the difference equations?

- IN PRINCIPLE the matrix method still works, also for difficult cases
  - we have system of ODE
  - we can write it as Cn = M Co
  - we can determine the eigenvalues
- are there simple cases with a simple formula?
- we keep to simple, constant coefficient cases for now
- we use simple BC, namely periodic
  - all coefficients the same for all equations
  - no influence of BC in space, works for all equations
  - we can use Fourier decomposition (periodic, linear)
  - assumption of periodic BC allows simple trial solution
  - using this periodic trial solution is called Von Neumann analysis

1-D advection-diffusion explicit with CD for advection constant U,V

$$\frac{Cnew(i)-Cold(i)}{\Delta t} = -U\frac{Cold(i+1)+Cold(i-1)}{2\Delta x} + K\frac{Cold(i+1)-2Cold(i)+Cold(i-1)}{\Delta x^2}$$

Equivalently:

$$\frac{C^{n+1}(i)-C^{n}(i)}{\Delta t} = -U\frac{C^{n}(i+1)+C^{n}(i-1)}{2\Delta x} + K\frac{C^{n}(i+1)-2C^{n}(i)+C^{n}(i-1)}{\Delta x^2}$$

The discretisation considered: 1-step methods

Remember that explicit schemes amounted to an equation of this form:

$$C^{n+1}(i) = a(i)C^{n}(i-1) + b(i)^{n}C(i) + c(i)^{n}C(i+1)$$

An implicit scheme gave rise to an equation of this form:

$$a(i)C^{n+1}(i-1) + b(i)C^{n+1}(i) + c(i)C^{n+1}(i+1) = C^{n}(i)$$

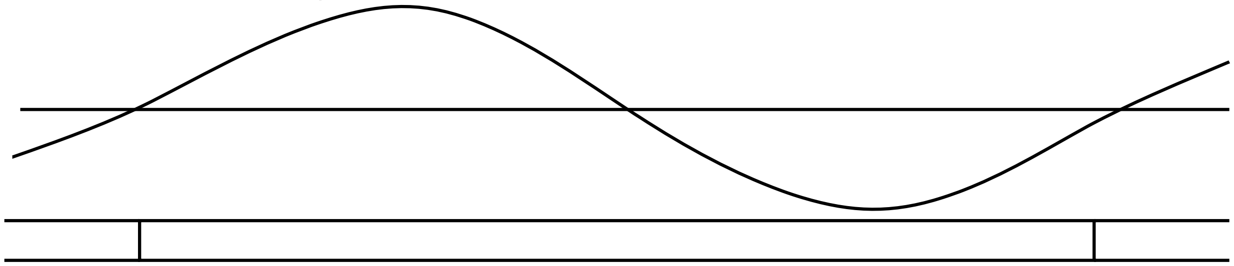
More generally, we have:

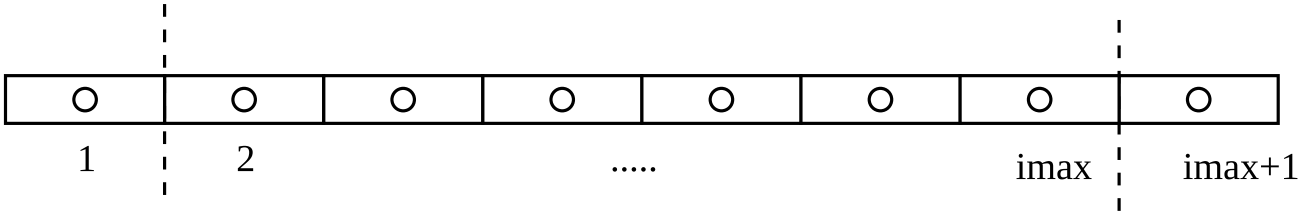
$$\begin{aligned} aL(i)C^{n+1}(i-1) + bL(i)C^{n+1}(i) + cL(i)C^{n+1}(i+1) = \\ aR(i)C^{n}(i-1) + bR(i)C^{n}(i) + cR(i)C^{n}(i+1) \end{aligned}$$

Taking constant coefficients:

$$\begin{aligned} aL * C^{n+1}(i-1) + bL * C^{n+1}(i) + cL * C^{n+1}(i+1) = \\ aR * C^{n}(i-1) + bR * C^{n}(i) + cR * C^{n}(i+1) \end{aligned}$$

Periodic BC:





$$C(1) = C(imax)$$
$$C(imax+1) = C(2)$$
$$L = x(imax) - x(1)$$

Analytic periodic:

$$sin(k*2*\pi*x/L)$$
$$cos(k*2*\pi*x/L)$$
$$e^{j*k*2*\pi*x/L}$$

Analytic periodic:

$$sin(k*2*\pi*x(i)/L)$$
$$L=(imax-1)*\Delta x$$
$$x(i)=(i)*\Delta x$$
$$sin(k*2*\pi*i/(imax-1))$$
$$sin(\alpha*i)$$
$$e^{j*\alpha*i}$$

The variation in i-direction is assumed to be of the form

$$e^{j*\alpha*i}$$

The variation in time is assumed to be of the "growth" form: λ^n. Together we get the trial solution:

$$\lambda^n e^{j*\alpha*i}$$

Substitute $\lambda^n e^{j*\alpha*i}$ in the general form:

$$\begin{aligned} &aL*C^{n+1}(i-1)+bL*C^{n+1}(i)+cL*C^{n+1}(i+1)= \\ &\qquad aR*C^{n}(i-1)+bR*C^{n}(i)+cR*C^{n}(i+1) \end{aligned}$$

We get

$$\begin{aligned} &aL*\lambda^{n+1}e^{j*\alpha*(i-1)}+bL*\lambda^{n+1}e^{j*\alpha*i}+cL*\lambda^{n+1}e^{j*\alpha*(i+1)}= \\ &\qquad aR*\lambda^{n}e^{j*\alpha*(i-1)}+bR*\lambda^{n}e^{j*\alpha*i}+cR*\lambda^{n}e^{j*\alpha*(i+1)} \end{aligned}$$

Divide both sides by the common factor $\lambda^n e^{j*\alpha*i}$.

$$aL * \lambda e^{j*\alpha*(-1)} + bL * \lambda + cL * \lambda * e^{j*\alpha*(+1)} =$$
$$aR * e^{j*\alpha*(-1)} + bR + cR * e^{j*\alpha*(+1)}$$

Resulting in the following expression for λ

$$\lambda = \frac{aR * e^{-j*\alpha} + bR + cR * e^{j*\alpha}}{aL * e^{-j*\alpha} + bL + cL * e^{j*\alpha}}$$

How can we use this expression?

Two examples: 1-D diffusion, 1-D advection

1-D diffusion. Scheme is

$$\frac{Cnew(i) - Cold(i)}{\Delta t} = +K\frac{Cold(i+1) - 2Cold(i) + Cold(i-1)}{\Delta x^2}$$
$$Cnew(i) = Cold(i) + \Delta t K\frac{Cold(i+1) - 2Cold(i) + Cold(i-1)}{\Delta x^2}$$
$$Cnew(i) = Cold(i) + D(Cold(i+1) - 2Cold(i) + Cold(i-1))$$
$$C^{n+1}(i) = C^n(i) + D(C^n(i+1) - 2C^n(i) + C^n(i-1))$$

with $D = \frac{\Delta t K}{\Delta x^2}$. We have:

$$aL = 0$$
$$bL = 1$$
$$cL = 0$$
$$aR = D$$
$$bR = 1 - 2D$$
$$cR = D$$

$$\lambda = \frac{aR * e^{-j*\alpha} + bR + cR * e^{j*\alpha}}{aL * e^{-j*\alpha} + bL + cL * e^{j*\alpha}} =$$

$$\frac{D * e^{-j*\alpha} + 1 - 2D + D * e^{j*\alpha}}{1} =$$

$$D * e^{-j*\alpha} + 1 - 2D + D * e^{j*\alpha} =$$

$$1 - 2D + D * (e^{-j*\alpha} + e^{j*\alpha}) =$$

$$1 + 2D(cos(\alpha) - 1)$$

stability if

$$-1 \leq 1 + 2D(cos(\alpha) - 1) \leq 1$$

Only relevant ($cos(\alpha) - 1$ is always negative):

$$\begin{aligned} -1 \leq 1 + 2D(cos(\alpha) - 1) \\ 2D(1 - cos(\alpha)) \leq 2 \\ D \leq \frac{1}{(1 - cos(\alpha))} \\ D \leq 2 \rightarrow \frac{\Delta t K}{\Delta x^2} \leq 2 \end{aligned}$$

EXERCISE

Show that the following scheme is always instable.

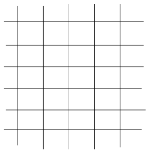
$$\frac{Cnew(i) - Cold(i)}{\Delta t} = -U\frac{Cold(i+1) - Cold(i-1)}{2\Delta x}$$

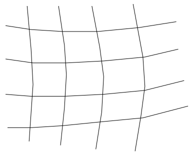
$$\frac{C^{n+1}(i) - C^{n}(i)}{\Delta t} = -U\frac{C^{n}(i+1) - C^{n}(i-1)}{2\Delta x}$$

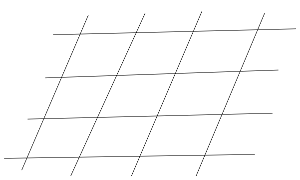
$$\lambda = \frac{aR * e^{-j*\alpha} + bR + cR * e^{j*\alpha}}{aL * e^{-j*\alpha} + bL + cL * e^{j*\alpha}}$$

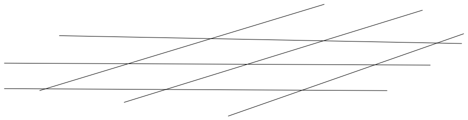
grid quality

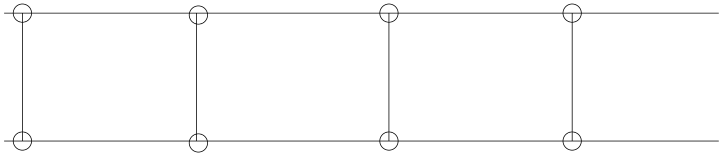
Quads

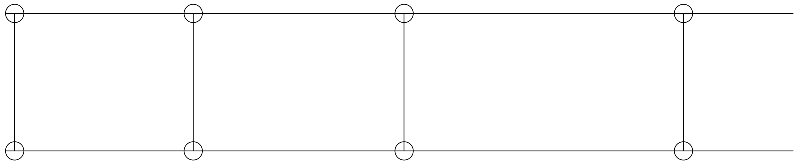


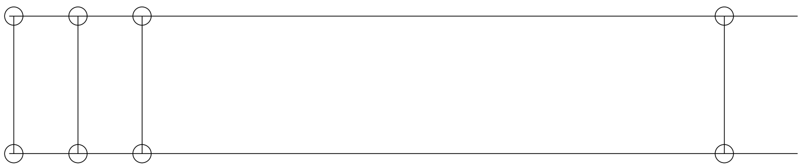




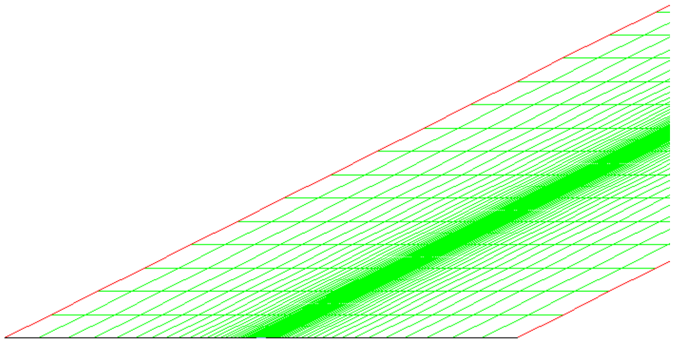






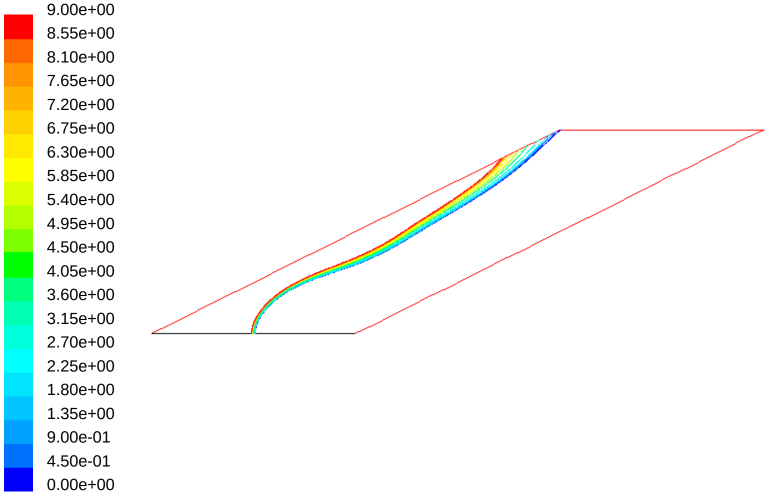


- angles close 90 degrees (larger then 20)
- stretch between 1 and 1.3
- ratio length/width between 0.1 and 10 (except close to wall)



Grid

May 18, 2005
FLUENT 6.1 (2d, dp, segregated, lam)



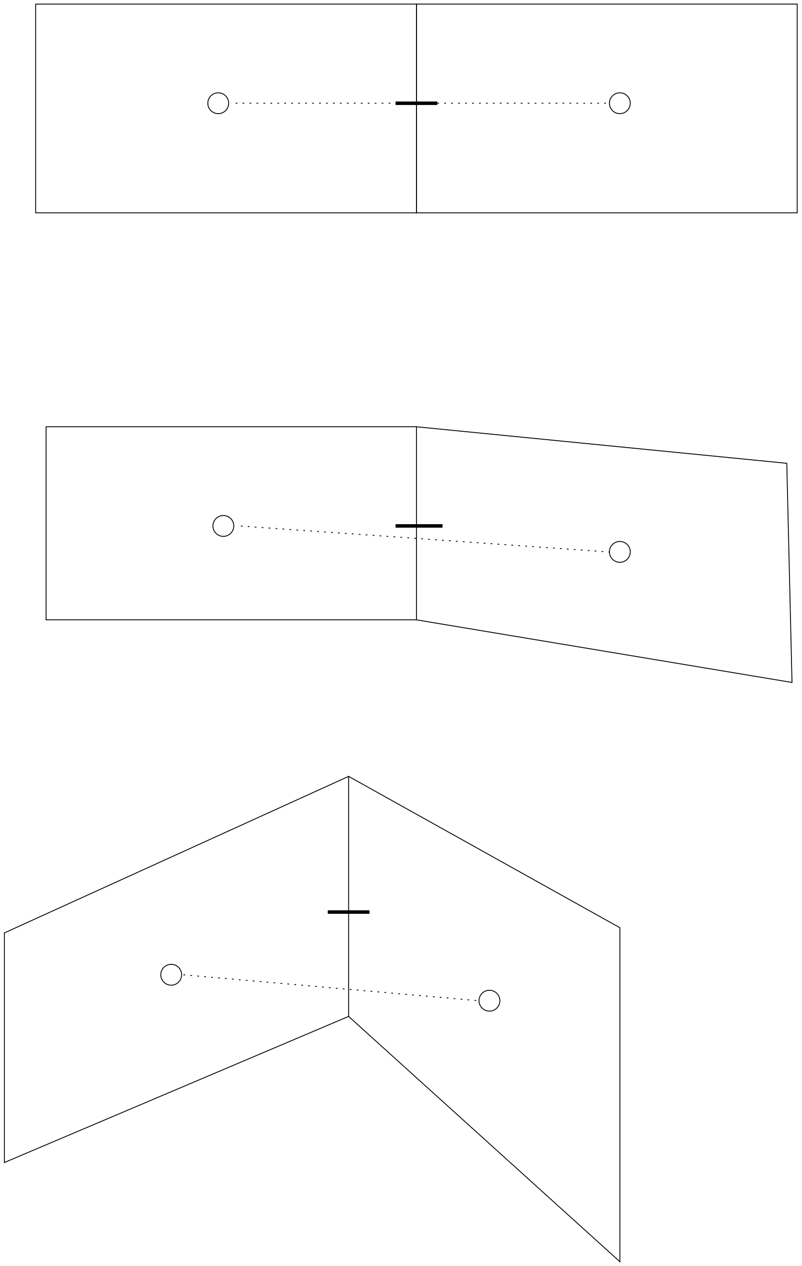
Path Lines Colored by Particle ID

May 18, 2005
FLUENT 6.1 (2d, dp, segregated, lam)

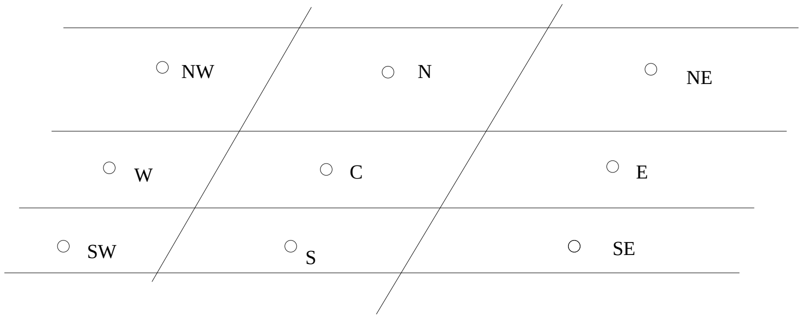
Difficulties with distorted grids:

- less accurate
- slower convergence

interpolation less accurate

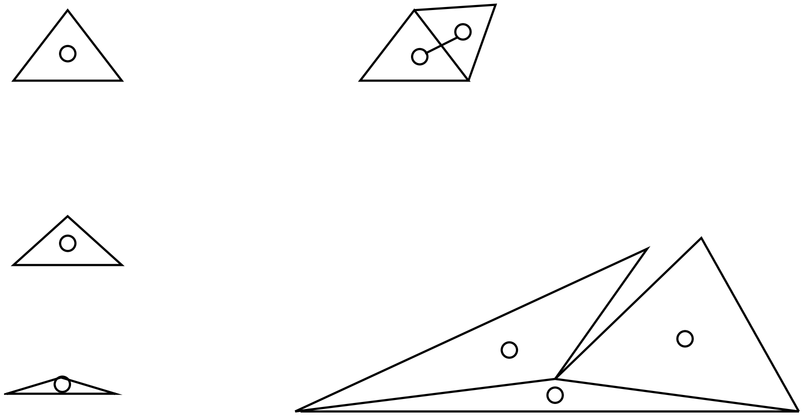


convergence: treatment of diagonal neighbors

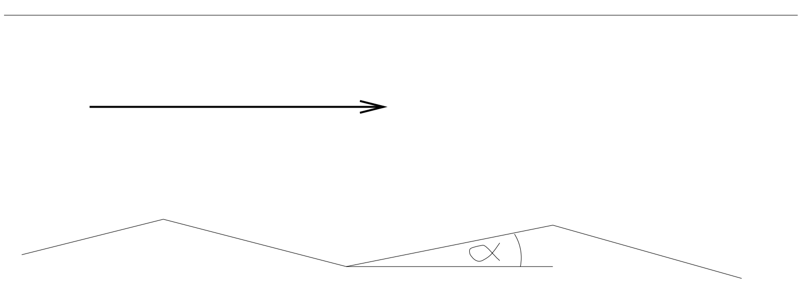


grid quality

Triangles

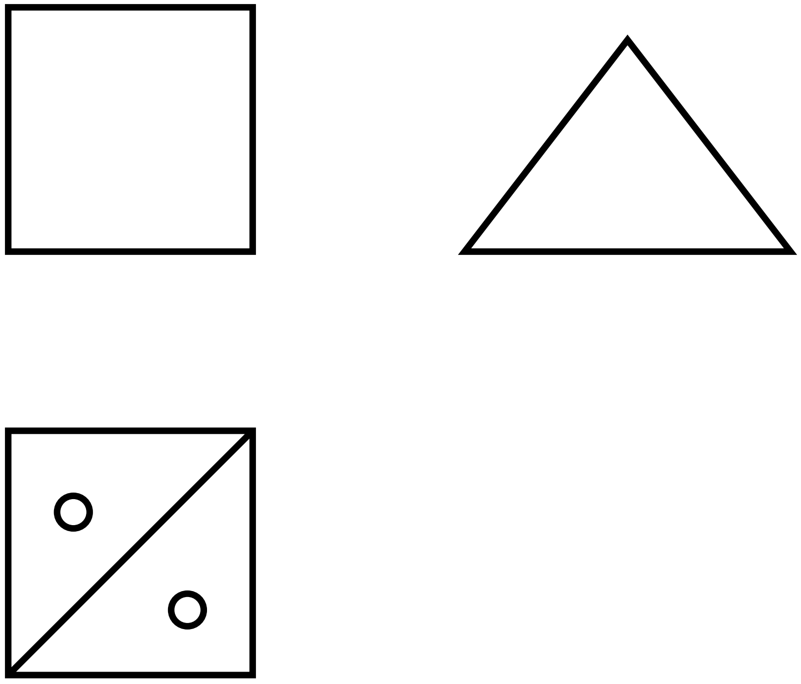


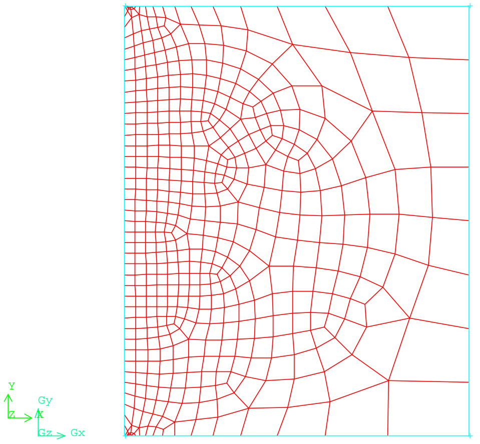
Sometimes compromise necessary: shallow riblet on surface

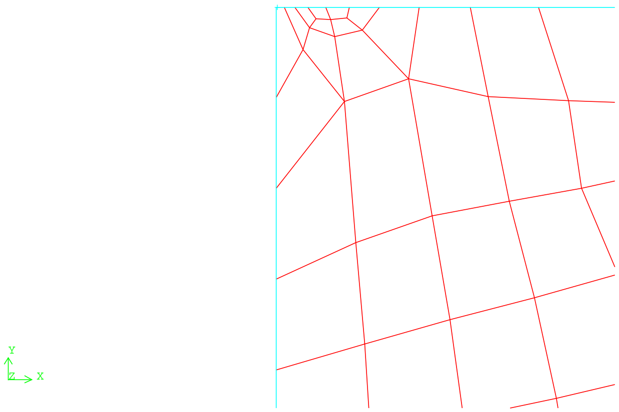


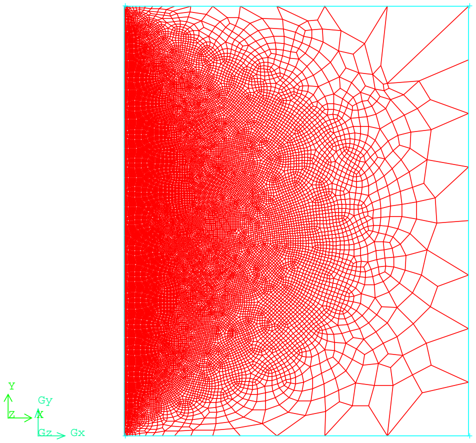
Grid tips (get the document when you start the assignment!)

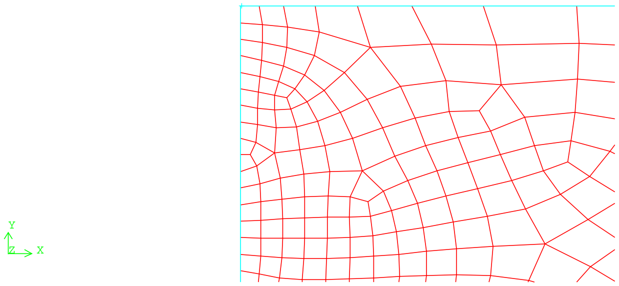
- Near a wall try to have quads
  - Follow streamline
  - Less cross-wind diffusion
  - Cheaper
- Gradual (smooth) transition of cell side, cell volume etc
- Nice looking grids are good grids




Y
Z
X
Gy
Gz
Gx


Y
Z
X


Y
Gy
Z
X
Gz
Gx


Y
Z
X

Sometimes a compromise is necessary

- lack of time
- very difficult
- ugly part in place with small change of gradient
  - NOT near wall
  - NOT near relevant phenomena (vortices, shear layers)
  - especially NOT near separation points
- airfoil: not too close to the airfoil

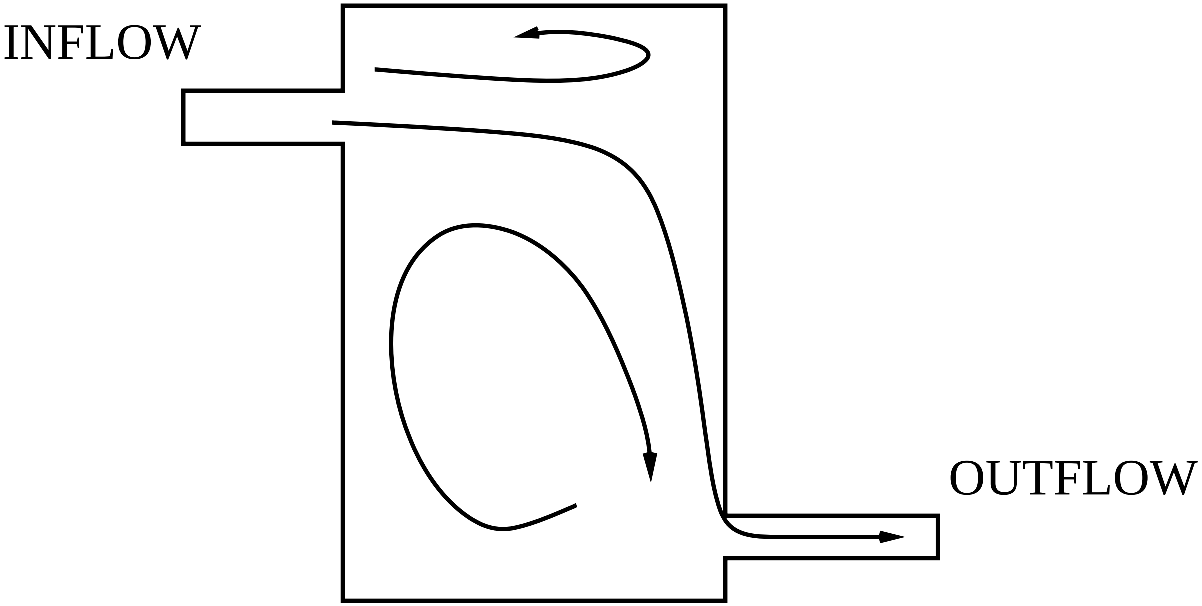

BROM

Boundary conditions in Fluent

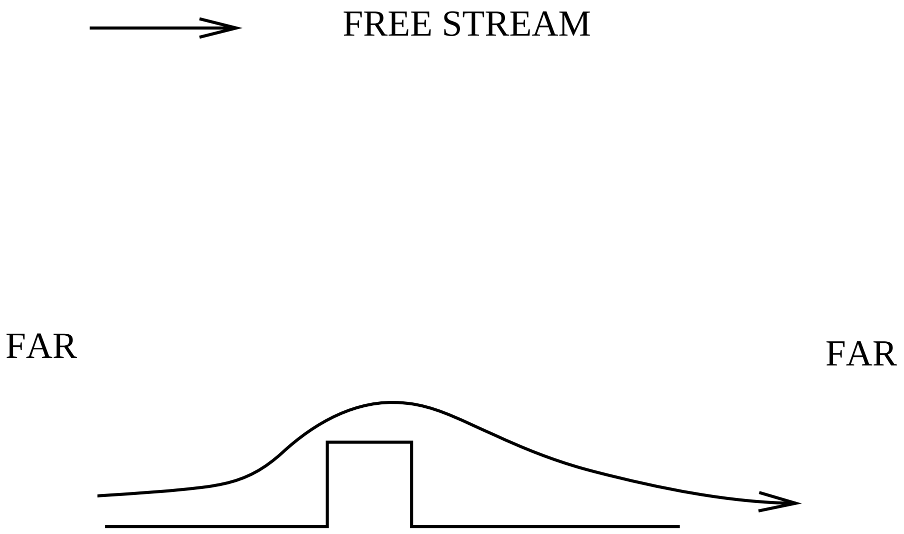
- inflow
- wall
- outflow
- periodic
- pressure
- symmetry
- on a rectangular grid, all BC amount to prescribing variable or its derivative

Some sample geometries

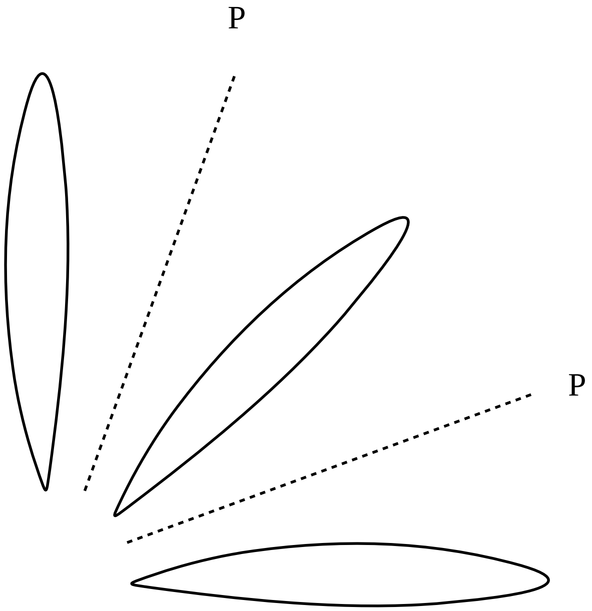
Interior flow



Exterior flow



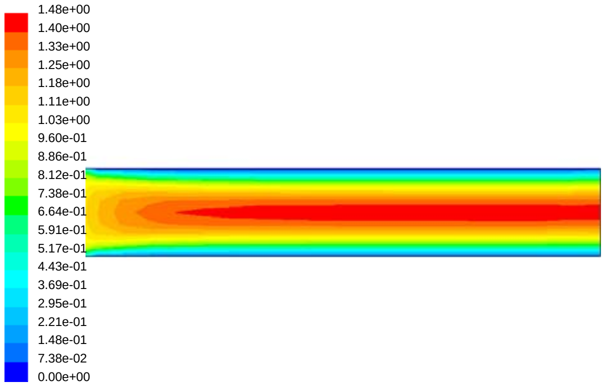
Periodic flow



Inflow

- prescribe quantities flowing in
- velocity components
- concentration, temperature
- where do you put the inflow boundary?
  - velocity etc well known
  - velocity etc NOT well known: let flow develop
- flow needs distance to become fully developed!
- in a straight channel, pipe: check pressure, normal velocity

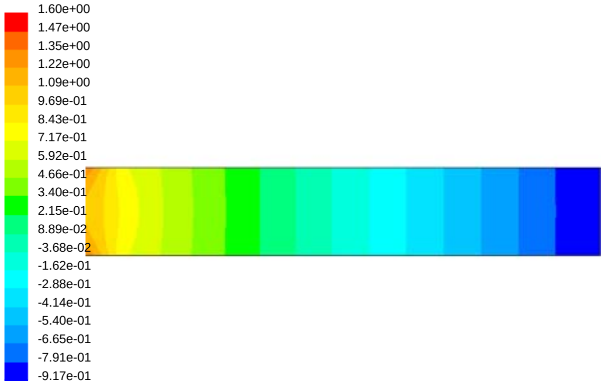
x-velocity



Contours of X Velocity (m/s)

May 17, 2005
FLUENT 6.1 (2d, dp, segregated, lam)

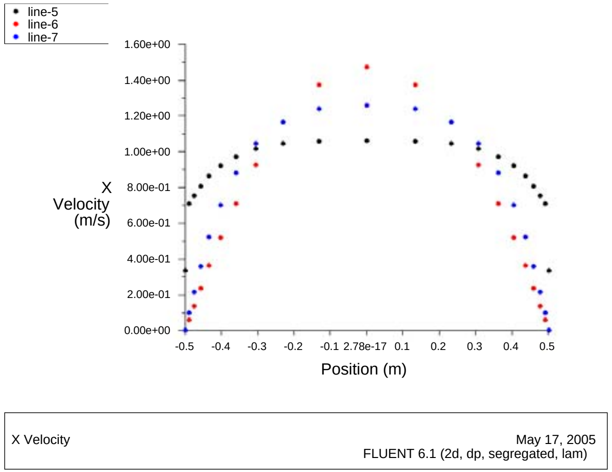
pressure



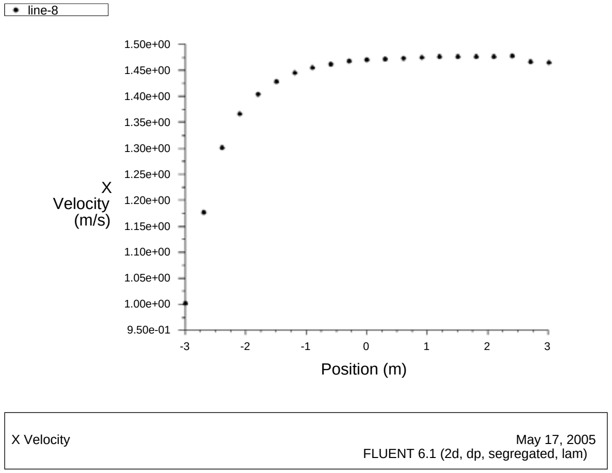
Contours of Static Pressure (pascal)

May 17, 2005
FLUENT 6.1 (2d, dp, segregated, lam)

Inlet x-velocity



x-velocity centerline



Entrance lenghts, see White p 331

laminar

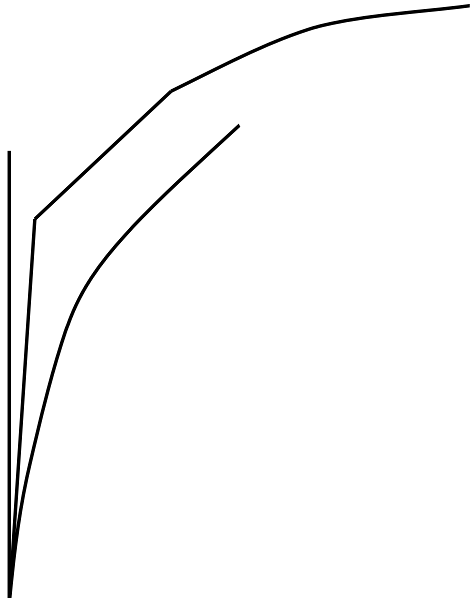
$$0.06ReD$$

turbulent

$$4.4Re^{\frac{1}{6}}D$$

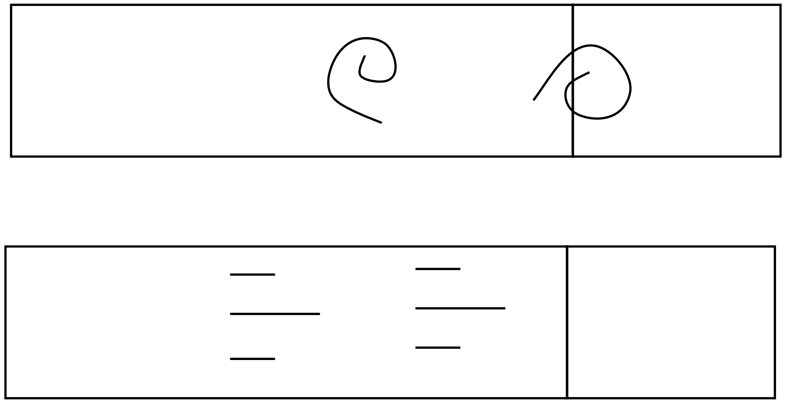
Wall

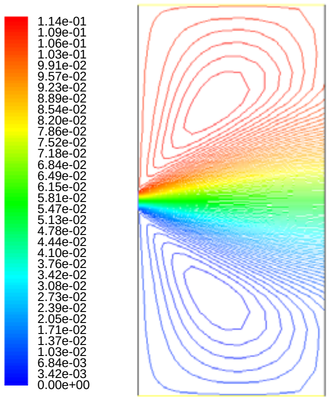
- velocity is 0
- all velocity components 0
- mind the grid: solution changes very much here!
- check for kinks in the results
- temperature:
  - prescribe temperature
  - prescribe heat flux
  - exchange coefficient



Outflow

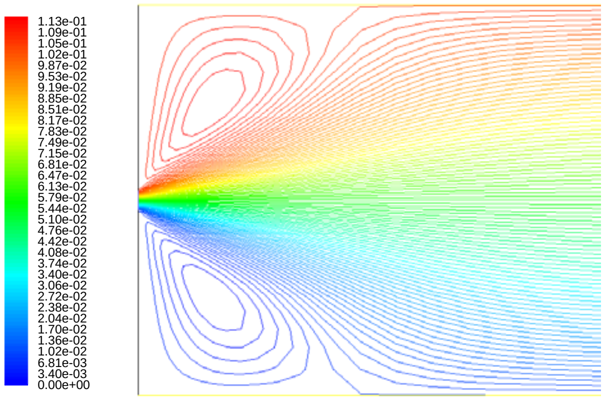
- cut calculation domain
- guess how solution continues
- commercial CFD: derivative 0
- ONLY valid for fully developed flow!
- avoid backflow (convergence problems, inaccuracy)
- add some extra domain if you have to





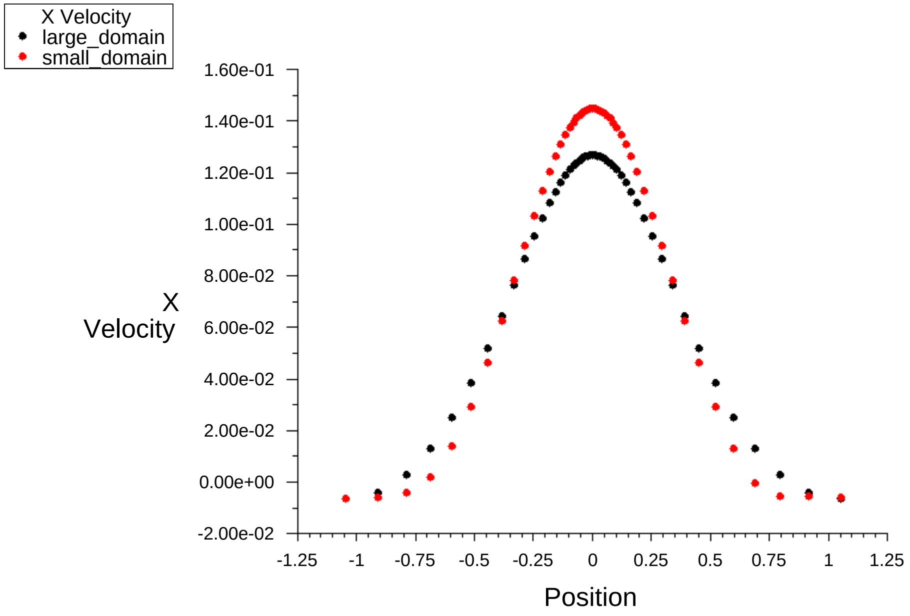
Contours of Stream Function (kg/s)

May 28, 2008
FLUENT 6.3 (2d, dp, pbns, lam)



Contours of Stream Function (kg/s)

May 28, 2008
FLUENT 6.3 (2d, dp, pbns, lam)



X Velocity

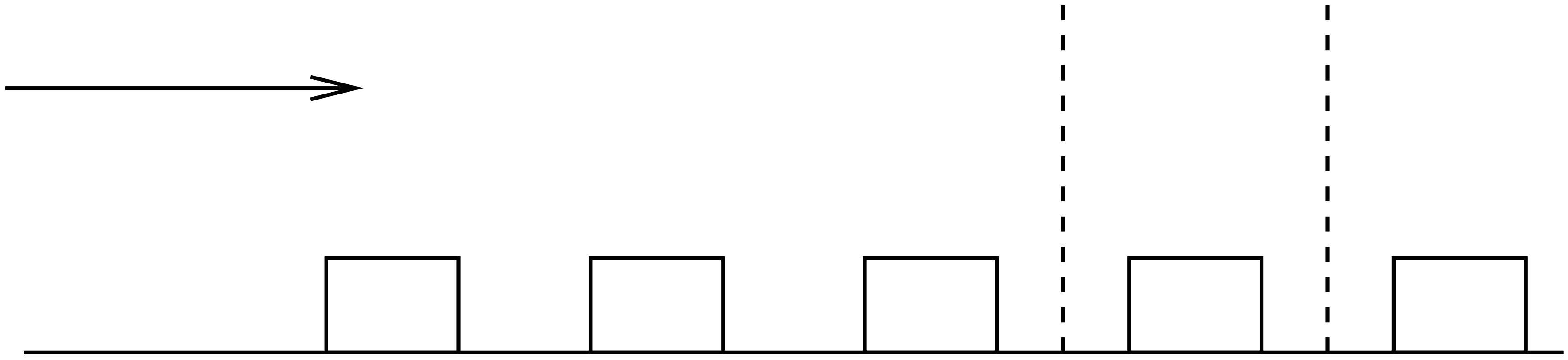
May 28, 2008
FLUENT 6.3 (2d, dp, pbns, lam)

Periodic

- vanes in turbine
- many houses in city
- nice for mathematical investigation
- NO extra energy added or removed
- ALL types of equations
- mathematically convenient

Example

- flow over lots of obstacles



Pressure outflow

- pressure prescribed
- velocity
  - normal to boundary
  - neighboring cell
  - prescribed direction
- open boundary conditions (entrainment)

Symmetry

- normal velocity is 0
- tangential velocity components derivative 0
- symmetry
- stable far field condition (but no entrainment)

- In principle two assignments, one fluent, one matlab
- work out solution
- report
  - relevant plots
  - explain what you did
  - tell/summarise everything you did (checking, other things you tried)
  - MUST be hand-written
  - in the form of short statements
  - hand-written, numbered figures
  - discussion with mentor
  - finish within two months
  - QUANTITATIVE comparison
    - decide which quantities are relevant/sensitive
    - 1-D plots matlab/fluent in one figure
    - numbers
    - only contour plots NOT sufficient

After this:

- discussion with the mentor

A quizz

- Preferably around the same time as the assignment
- Small errors in algebra should not matter
- You should be able to explain HOW to do something to a colleague student
  - stability analysis
  - order estimate
  - integral balance
  - discretisation
  - grid quality